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1990 J. Phys. A: Math. Gen. 23 5667

(http://iopscience.iop.org/0305-4470/23/23/036)

J. Phys. A: Math. Gen. 23 (1990) 5667-5671. Printed in the UK

COMMENT

Comment on 'Supercritical effects and the delta potential'

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Received 4 September 1990

Abstract. Recently Loewe and Sanhueza examined supercritical effects caused by a δ potential acting on a Dirac particle in one dimension and also by a δ-shell potential in three dimensions. Based on the observation that supercritical effects are absent for the δ potential, they suggested that the effects depend on the spatial extension of the potential. We point out that the effects are absent for a class of non-local separable potentials in one dimension. The ranges of the potentials can be chosen arbitrarily; the δ potential is a special case.

In a recent paper entitled 'Supercritical effects and the δ potential' Loewe and Sanhueza [1] (referred to as LS below) examined the possibility of supercritical effects induced when the Dirac sea is perturbed by an external potential. These supercritical effects occur when the potential becomes so strong that a bound Dirac particle dives into the negative energy sea. LS examined two models, one in one dimension and the other in three dimensions. In the one-dimensional model they assumed an attractive δ-function potential

$$V(x) = -g\delta(x) \tag{1}$$

where g is the strength parameter which corresponds to the a of LS. In the three-dimensional model they assumed a δ-shell potential

$$V(r) = -g\delta(r - r_0) \tag{2}$$

where r_0 is the shell radius. We use natural units ($c = \hbar = 1$). For the Dirac matrices we use the usual ones as LS did. In both models V is taken as the zeroth component of a Lorentz vector.

LS observed that the energy eigenvalue E of the positive parity bound state of a Dirac particle in one dimension does not reach $-m$ for any finite value of g, and hence no supercritical effects occur. However, the E of the three-dimensional model does reach $-m$ for a finite value of g. In their conclusions LS state that 'it is clear that these supercritical effects depend on the spatial extension of the potential'. By this statement they seem to imply that the zero-range nature of potential (1) is responsible for the absence of supercritical effects.

The purpose of this comment is to point out that, contrary to LS's statement quoted above, the spatial extension of the potential has no relevance to the absence of supercritical effects. To this end we examine a class of non-local separable interactions in one dimension, of which the δ potential is a special case, and show that supercritical effects do not occur, irrespective of the range of the separable potential. We also discuss the three-dimensional Dirac equation with a class of non-local separable potentials, of which the δ-shell potential is a special case.

Let us start with the one-dimensional case. There is an arbitrariness in defining the δ potential in the one-dimensional Dirac equation [2, 3]. The Dirac equation can be integrated by using

$$\int \mathrm{d}x\, \delta(x)\psi(x) = [\psi(0+) + \psi(0-)]/2 \tag{3}$$

where $\psi(x)$ is the wavefunction and 0+ stands for 0 plus positive infinitesimal. Then the eigenvalue of the positive parity state is given by [4]

$$E = m\left(\frac{4-g^2}{4+g^2}\right). \tag{4}$$

There is another bound state, with odd parity, and its energy is given by the negative of the E of (4). Throughout this comment, however, we focus on the positive parity bound state of the lowest energy. The E of (4) becomes $-m$ only in the limit of $g \to \infty$.

The potential V of (1) together with (3) can be regarded as a special case of the non-local separable potential defined by

$$V\psi = -gv(x) \int_{-\infty}^{\infty} \mathrm{d}x'\, v(x')\psi(x') \tag{5}$$

where $v(x) = v(-x)$. The V of (1) is obtained by $v(x) \to \delta(x)$. The Dirac equation with potential (5) has been examined in [3]. Note that h of [3] corresponds to our g, while the g (for the Lorentz scalar potential) of [3] is set to be zero. For the V of (5) one obtains

$$E = m(4 - g^2\mathscr{I}^2)/(4 + g^2\mathscr{I}^2) \tag{6}$$

where

$$\mathscr{I} = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}x'\, \mathrm{e}^{-\kappa|x-x'|} v(x)v(x') \tag{7}$$

and $\kappa = (m^2 - E^2)^{1/2}$. It is understood that the integral $\mathscr{I}$ is well defined and finite. Again $E \to -m$ only if $g \to \infty$. Actually (6) is still an involved equation for E because $\mathscr{I}$ depends on E through κ. If $v(x) \to \delta(x)$, then $\mathscr{I} \to 1$.

The solution of the one-dimensional Dirac equation for the square-well potential

$$V(x) = \begin{cases} 0 & \text{for} \quad |x| > r_0 \\ -D & \phantom{\text{for}} \quad |x| < r_0 \end{cases} \tag{8}$$

has been examined in detail, for example, by Greiner *et al* [5]. If one keeps r_0 fixed and increases the depth D, the energy eigenvalue E certainly reaches $-m$ at a finite value of D, hence supercritical effects occur. If one takes the 'δ-function limit' of the square-well potential, i.e. if one lets $r_0 \to 0$ and $D \to \infty$, with the condition

$$2r_0 D = g \tag{9}$$

one obtains

$$E = m \cos g. \tag{10}$$

Obviously $E \to -m$ at $g = \pi$ which is finite. Hence supercritical effects occur for this version of the δ potential. This is a counterexample of what LS hinted, i.e. these supercritical effects would not occur if the potential has no spatial extension. The results of the two versions of the δ-function potential, e.g. (4) and (10), are related to each other by the substitution

$$g \to 2\tan(g/2). \tag{11}$$

For the second version of the δ potential, (3) does not hold. This intriguing aspect has been discussed in [2, 3]; see also [6].

The Dirac wavefunction in one dimension has two components. Let the upper and lower components be ψ_1 and ψ_2, respectively. For the positive parity state which we are examining, ψ_1 and ψ_2 are even and odd functions of x, respectively. Note that both potentials defined by (1) and (5) do not act on ψ_2; this is because ψ_2 is an odd function of x. On the other hand the square-well potential (8) acts on ψ_2. One may suspect that this feature, i.e. whether or not the potential acts on ψ_2, is crucial to the absence of supercritical effects. This is not the case, however. In order to see this let us consider the following extension of the non-local potential

$$V\psi = -g\left[v_1(x) \int_{-\infty}^{\infty} dx'\, v_1(x')\psi(x') + v_2(x) \int_{-\infty}^{\infty} dx'\, v_2(x')\psi(x')\right] \quad (12)$$

where $v_1(x) = v_1(-x)$ and $v_2(x) = -v_2(-x)$. Note that the V of (12) acts on the odd function part of ψ through v_2. The Dirac equation with this interaction can be solved in a way similar to [3]. The bound-state energy E is determined by

$$[2\kappa + g(E+m)\mathscr{I}_{11}][2\kappa + g(E-m)\mathscr{I}_{22}] + \kappa^2 g^2 \mathscr{I}_{12}^2 = 0 \quad (13)$$

where

$$\mathscr{I}_{\mu\nu} = \mathscr{I}_{\nu\mu} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\, e^{-\kappa|x-x'|} v_\mu(x) v_\nu(x'). \quad (14)$$

If we assume that

$$v_2(x) = (x/|x|)v_1(x) \quad (15)$$

then $\mathscr{I}_{11} = \mathscr{I}_{22}$. In this special case, (13) can be reduced to

$$E = \frac{m[4 - g^2(\mathscr{I}_{11}^2 - \mathscr{I}_{12}^2)]}{\{[4 - g^2(\mathscr{I}_{11}^2 - \mathscr{I}_{12}^2)]^2 + 16g^2\mathscr{I}_{11}^2\}^{1/2}}. \quad (16)$$

It is not difficult to see from (16) that, in order for E to reach $-m$, g has to become infinite. In the general case in which $\mathscr{I}_{11} \neq \mathscr{I}_{22}$, the corresponding expression for E becomes complicated, but again one finds that $g \to \infty$ for $E \to -m$. The absence of supercritical effects (for a finite value of g) is characteristic of interaction (12); the spatial extension of the potential is irrelevant in this respect. Let us add that, in its zero-range limit, the square-well potential does not act on ψ_2. Nevertheless, supercritical effects occur as noted above.

Before ending our discussion of the one-dimensional case, let us mention the one-dimensional version of the δ-shell potential, i.e. the V of (2) used in one dimension with the understanding that $r = |x|$ or, more explicitly,

$$V(x) = -g[\delta(x - r_0) + \delta(x + r_0)]. \quad (17)$$

This δ-shell potential has a finite spatial extension. If the Dirac equation is integrated by using (3), one again finds that $E \to -m$ only if $g \to \infty$. Although it has two terms, potential (17) can be regarded as a special case of potential (12); this can be conveniently done by using partial waves in one dimension [7]. Hence the absence of supercritical effects is not surprising.

Let us now turn to the three-dimensional case. For the three-dimensional model with the δ-shell potential (2), LS solved the Dirac equation and found that the bound-state energy E reaches $-m$ for a finite value of g; hence supercritical effects occur. The δ-shell potential (2) can be rewritten in the form of

$$V\psi = -g\delta(r-r_0)r_0^{-2}\int \mathrm{d}\boldsymbol{r}'\,\delta(r'-r_0)\sum_{l,m}[Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi')]\psi(\boldsymbol{r}'). \tag{18}$$

This V acts on all partial waves equally. (The potential (17) can be regarded as a one-dimensional version of (18).) For the usual ground state with quantum number $j=\frac{1}{2}$, positive parity, the upper and lower components of the Dirac wavefunction are associated with the orbital angular momentum $l=0$ and 1, respectively. The potential (18) acts on the lower component as well as on the upper component.

Let us make two brief comments regarding the δ-shell potential.

I. The potential (18) can be generalized by replacing $\delta(r-r_0)$ with an arbitrary function $v_l(r)$; this is a three-dimensional version of potential (12). For such a potential the Dirac equation can be solved essentially in the same way as for the one-dimensional case by means of the Green function in coordinate space [3]. For an alternative method of solving the Dirac equation with a separable potential, see [8]. Regarding supercritical effects, such generalization does not seem to exhibit anything qualitatively different from those of the δ-shell potential. Therefore we will not go into details of the generalization.

II. Earlier we mentioned the arbitrariness in the definition of the one-dimensional δ-function potential [2, 3]. There is an exactly similar arbitrariness regarding the three-dimensional δ-shell potential. In replacing the potential (2) with the separable potential (18), it was assumed that (3) holds for the r-integration. Another interpretation of (2) is obtained by starting with a shell of a finite width, say, in the form of a square well; $V(r)=-D$ for $|r-r_0|<\varepsilon$ and $V(r)=0$ for $|r-r_0|>\varepsilon$. The Dirac equation can be solved without any ambiguity. One then takes the zero-range limit $\varepsilon\to 0$, keeping $2\varepsilon D=g$ fixed. It turns out that the results for this second version of the δ-shell potential can be obtained from those of the first version [with (3)] by the same substitution (11) as for the two versions of the one-dimensional δ-function potential. This substitution holds for any partial waves.

Finally let us add that the δ-shell potential in two dimensions is more like its three- (rather than one-) dimensional counterpart; see e.g. [9]. When g is increased starting from $g=0$, the eigenvalue E remains as $E=m$ (i.e. no bound state) for g less than a certain value, and E reaches $-m$ for another finite value of g, hence supercritical effects occur.

Acknowledgments

We would like to thank David Kiang for calling our attention to Loewe and Sanhueza's paper. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

Note added in proof. Supercritical effects are absent in certain examples in one dimension, but we know of no such examples in two and three dimensions. The spatial extension of one dimension is less than that of two and three dimensions. In this sense, we concur with Loewe and Sanhueza [1] who stated that "these supercritical effects depend on the spatial extension of the potential". We would like to thank Dr M Loewe for a helpful communication.

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